**Alternate Proof** Get to the point where we have

$$f_{n_i} = \sum_{r=1}^{i} g_r, \qquad \|g_r\| < \frac{1}{2^r}$$

then

$$\sum_{r=1}^{\infty} \int_X |g_r| d\mu < \infty$$

thus $\sum_{r=1}^{\infty} g_r$ converges pointwise almost everywhere. Let $f$ be the limit. It is defined almost everywhere. where it is not defined call it 0. We would like to show that $f_n \to f$ in $L^1$, in other words, show that the whole sequence converges there.

Given $\epsilon > 0$, $\exists n_0$ such that $n, m \geq n_0$ then

$$\int |f_m - f_n| d\mu < \epsilon$$

If we fix $n > n_0$ and let $m \to \infty$ then

$$\epsilon \geq \liminf \int |f_m - f_n| d\mu \geq \liminf |f_m - f_n| d\mu \geq \int |f - f_n| d\mu = \|f - f_n\|_{L^1}$$

all of this by fatous lemma. So we are done.

# 5 Hilbert Space and Completeness

Work in the abstract setting. $(X, \mathcal{F}, \mu)$ a measure space. Define

$$\mathcal{L}^2(X, \mathcal{F}, \mu) = \left\{ f : X \to \mathbb{C} | \text{measurable and } \int_X |f|^2 d\mu < \infty \right\}$$

and define $L^2$ as the equivalence classes

$$L^2(X, \mathcal{F}, \mu) = \{[f] | f \in \mathcal{L}^2(X, \mathcal{F}, \mu)\}$$

Easy to see that $\mathcal{L}^2$ vector space.

**Lemma.** *$f, g \in \mathcal{L}^2$ then $fg \in \mathcal{L}^1$*

*Proof.*

$$0 \le (f \pm g)^2 = |f|^2 \pm 2fg + |g|^2 \Rightarrow \pm 2fg \le |f|^2 + |g|^2$$

so

$$\int |fg| d\mu \le \int |f|^2 d\mu + \int |g|^2 d\mu$$

since both of the integrals on the right are finite, we are done. □

**Lemma.** $\mu(X) < \infty$ *implies that* $\mathcal{L}^2(X, \mathcal{F}, \mu) \subset \mathcal{L}^1(X, \mathcal{F}, \mu)$.

$\mathcal{L}^2$ is also a normed space with the norm

$$\|f\|_{L^2} = \left( \int_X |f|^2 d\mu \right)^{1/2}$$

its not hard to show that

1. $\|f\|_2 \ge 0$
2. $\|cf\|_2 = |c| \|f\|_2$
3. $\|f + g\| \le \|f\|_2 + \|g\|_2$ (Schwarz Inequality)

This space is quite different from $L^1$, because the norm comes from an **inner product**, which we define by

$$\langle f, g \rangle = \int_X f \overline{g} d\mu, \qquad L^2(X, \mu) \times L^2(X, \mu) \to \mathbb{C}$$

This inner product satisfies the following properties

1. $\langle f_1 + f_2, g \rangle = \langle f_1, g \rangle + \langle f_2, g \rangle$
2. $\langle cf, g \rangle = c \langle f, g \rangle$
3. $\overline{\langle f, g \rangle} = \langle g, f \rangle \Rightarrow \langle f, cg \rangle = \overline{c} \langle f, g \rangle$
4. $\langle f, f \rangle \ge 0$ and $\langle f, f \rangle = 0 \Rightarrow f = 0$

**Definition.** A pre-Hilbert space is a linear space $V$ with a sesquilinear inner product $V \times V \ni (v, w) \mapsto \langle v, w \rangle \in \mathbb{C}$.

This is a specific type of normed space since $\|v\| = \langle v, v \rangle^{1/2}$.

**Definition.** A **Hilbert Space** is a complete pre-Hilbert space.

We claim that $\|v\| = \langle v, v \rangle^{1/2}$. Not hard to prove, I omit it.

**Lemma.** *In an inner product space,* $|\langle v, w \rangle| \le \|v\| \|w\|$.

*Proof.*
$$0 \leq \langle v + \lambda w, v + \lambda w\rangle = \langle v, v\rangle + \lambda\langle w, v\rangle + \overline{\lambda}\langle v, w\rangle + \lambda^2\langle v, w\rangle$$

Choose
$$\lambda = -\frac{\langle v, w\rangle}{\langle w, w\rangle} \Rightarrow \|v\|^2 - 2\frac{|\langle v, w\rangle|^2}{\|w\|^2} + \frac{|\langle v, w\rangle|^2}{\|w\|^2} = \|v\|^2 - \frac{|\langle v, w\rangle|}{\|w\|^2}$$

so
$$|\langle v, w\rangle|^2 \leq \|v\|^2\|w\|^2 \Rightarrow |\langle v, w\rangle| \leq \|v\|\|w\|$$

□

We can use this to prove the triangle inequality by computing $\langle v + w, v + w\rangle$.

**Theorem.** *For $(X, \mathcal{F}, \mu)$ a measure space $L^2(X, \mu)$ is a Hilbert space*

Basically we want to show that its complete, with the norms

$$\|f\|_2 = \left(\int_X |f|^2 d\mu\right)^{1/2}, \qquad \langle f, g\rangle = \int_X f\overline{g}d\mu$$

*Proof.* Let $\{f_n\}$ be a Cauchy sequence, we will use the $L^1$ result. Set $Z = \{x \in X | f_n(x) \neq 0 \text{ for some n}\}$. Reason we do this is because if we set $Z_k = \{x \in X || f_n(x)| \geq 1/k\}$ measurable and of finite measure. The point is that

$$Z = \bigcup_{n,k}\{x \in X || f_n(x)| \geq 1/k\}$$

is measurable, and of finite measure, because (by Chebyshev)

$$\mu\{x \in X || f_n(x)| \geq 1/k\} \leq k^2 \int_X |f_n| d\mu$$

Now $Z = \bigcup_k Z_k$, $\mu(Z_k) < \infty$, $Z_{k+1} \supset Z_k \supset \ldots$. If $f \in L^2$, then $f|_{Z_k} \in L^1(Z_k, \mu)$ since

$$\int_{Z_k} 1|f| d\mu \leq (\mu(Z_k))^{1/2}\left(\int_X |f|^2 d\mu\right)^{1/2}$$

so $\{f_n|_{Z_k}\}$ Cauchy on $L^1(Z_k, \mu)$ $\forall k$. So we know that $f_n|_{Z_k} \to g_k \in L^1(Z_k, \mu)$ by completeness of $L^1(Z_k, \mu)$.

We showed $\exists$ a subsequence $f_{n(j)}|_{Z_k} \to g_k$ pointwise a.e. So we can extract successive subsequence $Z_1, \ldots, Z_k, \ldots$ so that $f_{n_{k(g)}} \to g_k$ pointwise on $L^1(Z_k, \mu)$ implies that we have a measurable function $g$ on $Z$ such that $f_{n_{k(g)}}(x) \to g$ on $Z_k$ pointwise. This converges to $g$, a.e. on $Z = \bigcup_{i=1}^{\infty} Z_k$. Modify on a set of measure 0 to get convergence everywhere, $g = 0$ on $X \setminus Z$.

$h_k$ is a subsequence of $f_n$ in $L^2(X, \mu)$. Consider the sequence $|h_m - h_n|$, $n$ fixed, $m$ variable, we know this is

1. sequence in $L^2(X,\mu)$
2. $|h_m(x) - h_n(x)|^2 \to |g(x) - h_n(x)|^2$ pointwise.

Fatou's Lemma implies

$$\int_X \liminf_m |h_m(x) - h_n(x)|^2 d\mu \leq \liminf_m \int_X |h_m - h_n|^2 d\mu$$
$$\int_x |g(x) - h_n(x)|^2 d\mu \leq \liminf_m \int_X |h_m - h_n|^2 d\mu$$

But given $\epsilon > 0$, $\exists N$ such that $m, n \geq N \Rightarrow \|h_n - h_m\|_2 \leq \epsilon$. So if $n \geq N$ we get

$$\|g - h_n\|_2 \leq \epsilon \Rightarrow h_n \to g \in L^2$$

□

Still in the setting of abstract Hilbert space, $H$ with inner product $\langle , \rangle$.

/